Int. J. Heat Mass Transfer. Vol. 5, pp. 521–524. Pergamon Press 1962. Printed in Great Britain.

ON A SYSTEM OF DIFFERENTIAL EQUATIONS FOR HIGHLY INTENSIVE HEAT AND MASS TRANSFER

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(*Received* 5 *August* 1961)

Аннотация—Показана возможность перехода от системы трех линейных дифференциальных уравнений высокоинтенсивного тепло-и массообмена (1–3) к обычным уравнениям теплопроводности Фурье (20–22) для комбинированной функции $(m_iT + n_i\theta + r_iP)$ температуры T, влагосодержания θ и избыточного давления P. Числа m_i, n_i и r_i выражаются через коэффициенты исходной системы (1–3).

NOMENCLATURE

- P, overpressure;
- θ, moisture content;
- T, temperature;
- ϵ, phase transformation criterion;
- Bu, Bulygin number;*
- Fo, Fourier number,
- Ko, Kossovich number;*
- Lu, Luikov number;*
- Pn, Posnov number.*

UNDER very intensive drying processes, transference of liquid and vapour in material takes place both under the influence of moisture content and temperature gradients and under the influence of an excess pressure gradient. Lebedev has determined by his experiments, for example, that, when drying wood by means of high-frequency currents and infra-red rays, about a half of all the moisture is removed as vapour under the influence of an excess pressure gradient. Analogous conditions are created under conductive–convective drying of paper and other materials. To calculate such processes of drying Luikov and Mikhailov [2] have proposed a system of three differential equations which, in a dimensionless critical form, can be written as

$$\frac{\partial T}{\partial Fo} = \nabla^2 T - \epsilon\, Ko \frac{\partial \theta}{\partial Fo}, \qquad (1)$$

$$\frac{\partial \theta}{\partial Fo} = Lu\nabla^2\theta - LuPn\nabla^2 T - Lu_P \frac{Bu}{Ko}\nabla^2 P, \qquad (2)$$

$$\frac{\partial P}{\partial Fo} = Lu_P\nabla^2 P + \epsilon \frac{Ko}{Bu}\frac{\partial \theta}{\partial Fo}. \qquad (3)$$

Here T, θ and P are dimensionless potentials of heat, mass and filtration transfer, respectively. We shall consider a general case when T, θ and P are the functions of independent dimensionless variables x, y, z and Fo. The other symbols used here are well known.

At present, only separate solutions of particular cases of this complex system have been published. We shall mention only an interesting solution given by Prudnikov [3] which was made for an analogous system. The analysis of this solution showed that the system of differential equations (1–3) can be reduced to a considerably simpler system of three common equations of heat conduction for the combined function $(mT + n\theta + rP)$ where m, n and r are numericals. For this purpose we shall apply the method used previously in [4, 5] for a system of two equations. Having determined $\nabla^2 T$ from equation (1) and $\nabla^2 P$ from (3) and using them in (2), we shall rewrite them as follows:

$$\nabla^2 T = \frac{\partial T}{\partial Fo} + \epsilon Ko \frac{\partial \theta}{\partial Fo}, \qquad (1')$$

$$\nabla^2\theta = \left[\frac{1}{Lu} + Pn\epsilon Ko - \frac{\epsilon}{Lu}\right] \times \frac{\partial \theta}{\partial Fo} + Pn\frac{\partial T}{\partial Fo} + \frac{Bu}{LuKo}\frac{\partial P}{\partial Fo}, \qquad (2')$$

* The definition of these criteria is given in [1].

$$\nabla^2 P = \frac{1}{Lu_P}\frac{\partial P}{\partial Fo} - \frac{\epsilon}{Lu_P}\frac{Ko}{Bu}\frac{\partial \theta}{\partial Fo}. \qquad (3')$$

Multiplying equations (1′), (2′) and (3′) respectively by the numbers m, n and r (which will be defined later), and adding them, we find:

$$\nabla^2 \{mT + n\theta + rP\} = \frac{\partial}{\partial Fo}\left\{(m + nPn)T + \left[n\left(\frac{1}{Lu} + Pn\epsilon Ko - \frac{\epsilon}{Lu}\right) + m\epsilon Ko - r\frac{\epsilon}{Lu_P}\frac{Ko}{Bu}\right]\theta + \left(n\frac{Bu}{LuKo} + \frac{r}{Lu_P}\right)P\right\}. \qquad (4)$$

Equation (4) is transformed into a common heat-conduction equation for the function $(mT + n\theta + rP)$ when the following conditions are fulfilled:

$$\frac{m + nPn}{m} = \frac{n\left(\frac{1}{Lu} + Pn\epsilon Ko - \frac{\epsilon}{Lu}\right) + m\epsilon Ko - r\frac{\epsilon}{Lu_P}\frac{Ko}{Bu}}{n} = \frac{n\frac{Bu}{LuKo} + \frac{r}{Lu_P}}{r} = \mu^2 \qquad (5')$$

or

$$1 + \frac{n}{m}Pn = \left(\frac{1}{Lu} + Pn\epsilon Ko - \frac{\epsilon}{Lu}\right) + \frac{m}{n}\epsilon Ko - \frac{r}{n}\frac{\epsilon}{Lu_P}\frac{Ko}{Bu} = \frac{n}{r}\frac{Bu}{LuKo} + \frac{1}{Lu_P} = \mu^2. \qquad (5)$$

Now let us find the numbers μ^2, m, n and r having expressed them by means of the coefficients of system (1–3). Eliminating n/m, n/r and taking inverse ratios from equation (5), we get a cubic equation for definition of μ^2:

$$\mu^6 - A\mu^4 + B\mu^2 - C = 0 \qquad (6)$$

$$A = 1 + \frac{1}{Lu_P} + Pn\epsilon Ko + (1 - \epsilon)\frac{1}{Lu} \qquad (7)$$

$$B = \left(1 + \frac{1}{Lu} + Pn\epsilon Ko\right)\frac{1}{Lu_P} + (1 - \epsilon)\frac{1}{Lu} \qquad (8)$$

$$C = \frac{1}{LuLu_P}. \qquad (9)$$

Substituting S for $\mu^2 - A/3$, equation (6) is transformed to

$$S^3 + Ds + E = 0 \qquad (10)$$

where

$$D = -\frac{A^2}{3} + B \qquad (11)$$

and

$$E = -\frac{2A^3}{27} + \frac{AB}{3} - C. \qquad (12)$$

The cube roots of equation (10) are defined by the formula

$$S = \sqrt[3]{\left[\left(-\frac{E}{2}\right) + \sqrt{\left(\frac{E^2}{4} + \frac{D^3}{27}\right)}\right]} + \sqrt[3]{\left[\left(-\frac{E}{2}\right) - \sqrt{\left(\frac{E^2}{4} + \frac{D^3}{27}\right)}\right]} = Z_1 + Z_2 \qquad (13)$$

where the following condition should be fulfilled:

$$Z_1 \cdot Z_2 = -\frac{D}{3}. \qquad (14)$$

Cubic equation (10) has three different real roots, for

$$\frac{E^2}{4} + \frac{D^3}{27} < 0. \qquad (15)$$

We shall consider only the case when the roots of the cubic equation are all real and different. Having calculated the roots of equation (10) and hence of equation (6) we shall find three numbers μ_1^2, μ_2^2 and μ_3^2. Using consecutively the numbers μ_1^2, μ_2^2 and μ_3^2 in equalities (5) and (5′) we shall get six independent equations for calculation of six ratios or nine unknown numbers. Assuming that m_1, n_2 and r_3 equal unity, we obtain

$$\left.\begin{aligned} m_1 &= 1 \\ n_1 &= \frac{\mu_1^2 - 1}{Pn} \\ r_1 &= \frac{(\mu_1^2 - 1)\, Lu_P Bu}{Pn Lu Ko\,(\mu_1^2 Lu_P - 1)} \end{aligned}\right\} \quad (16)$$

$$\left.\begin{aligned} m_2 &= \frac{Pn}{\mu_2^2 - 1} \\ n_2 &= 1 \\ r_2 &= \frac{Lu_P Bu}{(\mu_2^2 Lu_P - 1)\, LuKo} \end{aligned}\right\} \quad (17)$$

$$\left.\begin{aligned} m_3 &= \frac{Pn Lu Ko\,(\mu_3^2 Lu_P - 1)}{Lu_P Bu\,(\mu_3^2 - 1)} \\ n_3 &= \frac{(\mu_3^2 Lu_P - 1)\, LuKo}{Lu_P Bu} \\ r_3 &= 1. \end{aligned}\right\} \quad (18)$$

Using the values m_i, n_i, r_i and μ_i^2 found, equation (4), equivalent to system (1–3), may be written as follows:

$$\nabla^2\{m_i T + n_i\theta + r_i P\} = \mu_i^2 \frac{\partial}{\partial Fo}\{m_i T + n_i\theta + r_i P\} \quad (19)$$

where $i = 1, 2, 3$.

Assuming

$$K_{11} = \frac{1}{\mu_1^2}, \quad K_{22} = \frac{1}{\mu_2^2} \text{ and } K_{33} = \frac{1}{\mu_3^2},$$

we get the following final system of equations instead of (1–3):

$$\frac{\partial}{\partial Fo}\{T + n_1\theta + r_1 P\} = K_{11}\nabla^2\{T + n_1\theta + r_1 P\} \quad (20)$$

$$\frac{\partial}{\partial Fo}\{m_2 T + \theta + r_2 P\} = K_{22}\nabla^2\{m_2 T + \theta + r_2 P\} \quad (21)$$

$$\frac{\partial}{\partial Fo}\{m_3 T + n_3\theta + P\} = K_{33}\nabla^2\{m_3 T + n_3\theta + P\}. \quad (22)$$

Solving heat-conduction equations (20–22) under the boundary conditions of the first, second and third kind, we get the solution for the combined function $(m_i T + n_i\theta + r_i P)$.

Designate these solutions through $f_1\,(x, y, z, Fo)$, $f_2\,(x, y, z, Fo)$ and $f_3\,(x, y, z, Fo)$, respectively, then

$$T + n_1\theta + r_1 P = f_1 \quad (23)$$

$$m_2 T + \theta + r_2 P = f_2 \quad (24)$$

$$m_3 T + n_3\theta + P = f_3. \quad (25)$$

Each of the functions T, θ and P can be expressed directly by means of the functions f_1, f_2 and f_3.

Using for this purpose common and functional determinants, we obtain for the unknown functions T, θ and P the values

$$T = \frac{\Delta_T}{\Delta} \quad (26)$$

$$\theta = \frac{\Delta_\theta}{\Delta} \quad (27)$$

$$P = \frac{\Delta_P}{\Delta}, \quad (28)$$

where

$$\Delta = \begin{vmatrix} 1 & n_1 & r_1 \\ m_2 & 1 & r_2 \\ m_3 & n_3 & 1 \end{vmatrix} \neq 0 \qquad \Delta_T = \begin{vmatrix} f_1 & n_1 & r_1 \\ f_2 & 1 & r_2 \\ f_3 & n_3 & 1 \end{vmatrix}$$

$$\Delta_\theta = \begin{vmatrix} 1 & f_1 & r_1 \\ m_2 & f_2 & r_2 \\ m_3 & f_3 & 1 \end{vmatrix} \qquad \Delta_P = \begin{vmatrix} 1 & n_1 & f_1 \\ m_2 & 1 & f_2 \\ m_3 & n_3 & f_3 \end{vmatrix}.$$

The substitution of a complex system of differential equations (1–3) for three common heat-conduction equations of the same type has a great practical value. It is difficult to get a solution of a system of equations (1–3) even for a one-dimensional heat-transfer problem. These difficulties arise considerably when two- and three-dimensional problems of heat and mass transfer are solved.

At the same time, many solutions of one-, two- and three-dimensional heat-conduction problems for the most common boundary conditions are known. All these solutions with the above-mentioned method may be easily used for description and calculation of a number of heat- and mass-transfer processes.

The presentation of solutions in the form (26–28) is very convenient for finding and calculating the rate of heating of a material, the rate of moisture content and of excess pressure variations, as well as for determining their interconnexion.

REFERENCES

1. Y. A. Mikhailov, Highly intensive heat and mass transfer in dispersed media. *Int. J. Heat Mass Transfer*, **1**, 37–45 (1960).
2. A. V. Luikov and Y. A. Mikhailov, *Teoriya perenosa energii i veschestva*, Izd. AN BSSR, Minsk (1959). (*Energy and mass transfer theory*).
3. A. P. Prudnikov, K issledovaniyu teplo-i massoobmena v dispersnykh sredakh. *Inzh. Fiz. Zh.* **4**, 81–86 (1958). (On investigation of heat and mass transfer in disperse media.)
4. P. S. H. Henry, Diffusion in absorbing media, *Proc. Roy. Soc.* A **171**, 215–272 (1939).
5. M. S. Smirnov, O sisteme differentsialnykh uravnenii protsessa sushki. *Inzh. Fiz. Zh.* **9**, 40–44 (1961). (On a system of differential equations of a process of drying.)

Abstract—This paper shows the possibility of transition from a system of three linear differential equations of highly intensive heat and mass transfer (1–3) to ordinary Fourier heat conduction equations (20–22) for the combined function $(m_i T + n_i \theta + r_i P)$ of temperature T, moisture content θ and excess pressure P. The numbers m_i, n_i and r_i are expressed by the coefficients of the initial system (1–3).

Résumé—Cet article montre la possibilité de passer du système des trois équations différentielles linéaires du transport de chaleur et de masse intensif (1–3) aux équations de Fourier ordinaires de conduction thermique, pour la fonction composée $(m_i T + n_i \theta + r_i P)$ de la température T, de la teneur en eau θ, et de la pression P. Les nombres m_i, n_i et r_i sont exprimés par les coefficients du système initial (1–3).

Zusammenfassung—Für sehr intensiven Wärme- und Stoffübergang wird eine Möglichkeit gezeigt, das System der drei linearen Differentialgleichungen (1–3) in gewöhnliche, Fourier'sche Gleichungen für Wärmeleitung (20–22) überzuführen. Dazu wird eine kombinierte Funktion $(m_i T + n_i \theta + r_i P)$ der Temperatur T, des Feuchtigkeitsgehaltes θ und des Überdruckes P verwendet. Die Zahlen m_i, n_i und r_i sind durch die Koeffizienten des Anfangsystems (1–3) bestimmt.